Indistinguishability

Nick Huggett
Department of Philosophy, University of Illinois at Chicago

Tom Imbo
Department of Physics, University of Illinois at Chicago

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In the considerable physical and philosophical literature[1] 'indistinguishability', and the related concept of 'identicality', are used in many ways, and in the resulting confusion the logical relations between the various notions are often obscured, with unfortunate consequences. This article will use them in the following senses, which are most useful and (likely) common:

> Particles are *identical* if they share in common all their constant properties, such as mass, charge, spin and so on: that is, if they agree in all their state-independent or *intrinsic* properties. Particles are *indistinguishable* if they satisfy the indistinguishability postulate (*IP*). This postulate states that all observables O must commute with all particle permutations P: $[O, P] = 0$. Put informally, the IP is the requirement that no expectation value of any observable is affected by particle permutations.

The IP presupposes the following formal structure: assume that we have a system of n identical quantum particles, and that if n were equal to 1 then the state space of the system would be $\mathcal{H}_1$. The natural assumption for $n > 1$ is that the state space $\mathcal{H}$ describing the system is a subspace of the tensor product, $\mathcal{H}_n$, of n copies of $\mathcal{H}_1$. That is,

$$\mathcal{H} \subseteq \mathcal{H}_n \equiv \bigotimes_{i=1}^{n} \mathcal{H}_1. \tag{1}$$

We assume that $\mathcal{H}$ is closed under the action of arbitrary permutations, P, which permute the n factors of $\mathcal{H}_n$. Any such operator is a product of 'particle exchange operators' P_{ij} $(1 \leq i, j \leq n)$. P_{ij} interchanges the ith and jth copies of $\mathcal{H}_1$ in $\mathcal{H}_n$: for instance (for $n = 2$),

$$P_{12}(|\phi\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\phi\rangle. \tag{2}$$

[1] See [1] for a cross section of the philosophical literature, and a comprehensive bibliography of the subject

For example, if the particles are either bosons or fermions then the appropriate state spaces are the symmetric ($P_{ij}|\Psi\rangle = |\Psi\rangle$) and antisymmetric ($P_{ij}|\Psi\rangle = -|\Psi\rangle$) subspaces of $\mathcal{H}_n$ respectively. Operators that commute with all permutations are called *symmetric*. The IP says that only symmetric Hermitian operators are observables; any non-symmetric Hermitian operators on $\mathcal{H}$ do not correspond to observable quantities if the IP holds.

1 Logical Relations

Oftentimes (e.g., [2], 275-6) an attempt is made to connect identicality and indistinguishability by appeal to the fact that in quantum mechanics (QM), unlike classical mechanics, particles cannot have varying continuous trajectories. Even if a particle has a definite location at some times, its position will be indefinite at times in between. Why? States of definite position – eigenstates of position – are necessarily orthogonal, and it is impossible for a system to occupy a continuous series of orthogonal states. (Any unitary evolution between such states will take a finite time, and under measurement the probability of collapse to an orthogonal state is zero.) And of course there is nothing special about position in this regard: even if the spectrum of an operator is continuous, no quantum evolution corresponds to a continuous trajectory through the spectrum.

This line of thought is supposed to lead directly to the conclusion that identical quantum particles (unlike classical particles) cannot be distinguished by continuous trajectories (through space or the spectrum of any observable). So there are two questions: (i) Does this conclusion – call it *trajectory indistinguishability* – actually follow? (ii) What do these considerations have to do with indistinguishability as defined earlier?

1.1 Trajectory Indistinguishability

First (i). This argument is supposed to show that quantum particles are trajectory indistinguishable, *without appeal to the IP* (from which it follows immediately, as discussed below). The idea behind the argument is that quantum particles can only be distinguished by continuous trajectories that are constant – because, as we just saw, varying continuous trajectories are impossible. But the identicality of the particles is supposed to preclude their being distinguished by constant properties. However, there is a fallacy in this line of thought. A property is 'intrinsic' if it is independent of any *possible* state of the system, not simply if it is a constant of some particular evolution; so identical particles *can* be distinguished by constant trajectories.

For instance, let $\mathcal{H}_1$ be a 2-dimensional Hilbert space spanned by $\{|\lambda_1\rangle, |\lambda_2\rangle\}$, eigenstates of the time-independent observable A with eigenvalues λ_1 and λ_2, respectively. Further suppose that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_1$, and that all Hermitian operators are observables and indeed allowed Hamiltonian operators. Then one possible evolution of the system is $\Psi(t) = |\lambda_1\rangle \otimes |\lambda_2\rangle$ (for all t), in which the particles are distinguished by their constant 'trajectories' – the first always has the value λ_1 for A and the other λ_2.[2] But the values of A are not state independent: there are states in $\mathcal{H}$ in which the value of A for the first particle is not λ_1 (for instance, $|\lambda_2\rangle \otimes |\lambda_1\rangle$), and states in which the particles have no definite A value (for instance, $a|\lambda_1\rangle \otimes |\lambda_2\rangle + b|\lambda_2\rangle \otimes |\lambda_1\rangle$). So A is not intrinsic, and indeed (supposing the particles do share their truly intrinsic properties) the example shows that identical particles can, after all, be trajectory distinguishable.

Note that in this example, the operators corresponding to the value of A for the two particles violate the IP, and hence their values would not constitute physical trajectories if the IP held. Indeed, although identical quantum particles are not necessarily trajectory indistinguishable, they will be if they are indistinguishable.[3]

1.2 Indistinguishability

In answer to (*ii*), the impossibility of continuously varying trajectories does not support indistinguishability in the sense of the IP. The IP is a constraint on which operators can be observables, but the impossibility of continuously varying trajectories is a fact about all Hermitian operators, whether or not they satisfy the IP. Hence this impossibility places absolutely no restriction on observables at all, once we adopt the quantum formalism.

Indeed, there are consistent (though hypothetical) quantum systems of identical particles that violate the IP: for instance, a collection of identical 'quantum Maxwell-Boltzmann' particles. For n such particles the state space is the full Hilbert space $\mathcal{H}_n$ of (1) – i.e., $\mathcal{H} = \mathcal{H}_n$ – and every (sufficiently well-behaved) Hermitian operator is an observable (as in the example above). Note that while this formalism is commonly used for non-identical particles, a system of n identical particles can also have $\mathcal{H}_n$ as its state space. Such particles are said to obey quantum Maxwell-Boltzmann or 'infinite' statistics.[4] This system clearly violates the IP, because some observables are non-symmetric: $[O, P] \neq 0$. In this sense then, the particles are 'distinguishable'.

While it is widely known, at least implicitly, that identicality does not imply the indistinguishability postulate, it seems rarely to be explicitly acknowledged, with certain resultant confusions about the nature of identical particles.[5] For

[2] A is not an operator on $\mathcal{H}$, so what is meant here is that $\Psi(t)$ is an eigenstate of $A \otimes I$ with eigenvalue λ_1, and of $I \otimes A$ with eigenvalue λ_2. That is, following the standard understanding, the operator 'corresponding' to A for the first particle is $A \otimes I$, and so on.

[3] It is often assumed that all single particle observables have the form $I \otimes \ldots \otimes I \otimes A \otimes I \ldots \otimes I$ (which violates the IP), but one might imagine a more general conception. What is essential, however, is that an observable representing a property of one particle be related by permutation to the observable representing the same property of another particle: as $A \otimes I$ and $I \otimes A$ are. But the IP means that permutations leave observables unchanged, in which case there cannot be a pair of *distinct* observables representing the same property for a pair of particles: hence no such pair of particles can have distinct trajectories.

[4] Such a system has second-quantized realizations whose particles are known as 'quons'. See [3] and references therein.

[5] Part of the confusion arises because 'identicality' is often used to mean indistinguishability. Although logically unproblematic, this usage obscures the possibility of particles that share their intrinsic properties, but violate the IP.

example, it seems that the 'problem of identical particles' is often taken to be the problem of understanding how the symmetrization postulate (*SP*) – that all particles are either bosons or fermions – can be shown to follow from the indistinguishability postulate, as if the latter were more secure than the former. (See, e.g., [4] for a discussion of the extra assumptions required by a derivation.). But there are no first principle grounds for holding indistinguishability either; certainly not as a logical consequence of quantum identicality. Thus both the symmetrization and indistinguishability postulates are on a very similar footing. As a matter of empirical fact, all known particles satisfy both, but no purely logical grounds exist for either. Indeed, the situation is that the SP entails the IP, but not the converse.[6] Thus, if one principle explains the other (and if entailment is a form of explanation), it is symmetrization that explains indistinguishability, not the other way around!

Of course, the fact that all known species of elementary particles are either bosons or fermions suggests that there may be some reason, some important principle, explaining why nature does not explore the many other options. Much work has been devoted to showing which additional principles are necessary to prove the IP or SP; but none of these principles seem more natural or secure than what is meant to be shown.

To summarize: It is important to keep clear the relations between the concepts of identicality, trajectory indistinguishability and indistinguishability (and symmetrization). First, identicality entails neither trajectory indistinguishability nor indistinguishability (though the former follows from the latter); the impossibility of continuously varying trajectories in QM is nothing but a red herring. Second, the SP implies the IP, but not the converse. So, to summarize the summary,

$$\text{SP} \Rightarrow \text{IP} \Rightarrow \text{Trajectory Indistinguishability}$$

but none of these follow from identicality.

2 Approximate Distinguishability

It is important to note that one can sometimes treat indistinguishable particles as 'approximately' distinguishable.

[6] An operator on $\mathcal{H}_n$ leaves the subspace of bosonic states, $\mathcal{H}_+$, invariant iff its action on $\mathcal{H}_+$ is the same as that of its projection onto $\mathcal{H}_+$; this latter operator necessarily satisfies the IP. Now, observables for a system of identical bosons must leave $\mathcal{H}_+$ invariant, else measurement collapses will not be well-defined. So not all Hermitian operators on $\mathcal{H}_n$ can be bosonic observables, only those whose action on $\mathcal{H}_+$ is the same as that of a symmetric operator; similarly for fermions, hence SP implies IP.

The SP can be derived from the conjunction of the IP and the assumption that the representation of the permutation group is 1-dimensional on $\mathcal{H}$: $P|\Psi\rangle = \lambda|\Psi\rangle$. The point is that there is no independent justification for the latter conjunct, which can be consistently relaxed, as we shall see in §3.

First, which properties are to count as intrinsic is a system-relative matter. Consider a system of two electrons that are in distinct *constant* spin-z eigenstates, one spin up and the other spin down, so that the spins function as intrinsic distinguishing properties for the particles. Now, this may seem surprising since the particles in question are identical fermions at a fundamental level, and hence their states are antisymmetric under the exchange operator P_{12}. Antisymmetrization (and, similarly, symmetrization for identical bosons) implies that the z-spins can never distinguish particle 1 – that is, the particle associated with the first 'slot' in the tensor product space – from particle 2 – the one associated with the second slot. For example, their state cannot be something like $|\uparrow\rangle \otimes |\downarrow\rangle \otimes |\psi\rangle$, in which particle 1 is the spin-up electron and particle 2 the spin-down electron, and $|\psi\rangle$ represents the non-spin portion of the two particle state. Suppose, however, that the Hilbert space of the system in question is spanned by states of the form

$$(|\uparrow\rangle \otimes |\alpha\rangle) \otimes (|\downarrow\rangle \otimes |\beta\rangle) - (|\downarrow\rangle \otimes |\beta\rangle) \otimes (|\uparrow\rangle \otimes |\alpha\rangle), \tag{3}$$

(in which, for example, the first term assigns spin-up and the non-spin state $|\alpha\rangle$ to particle 1, and spin-down and the non-spin state $|\beta\rangle$ to particle 2). Then we can 'distinguish' a spin-up particle from a spin-down particle in the following sense. In a state such as (3), $|\alpha\rangle$ ($|\beta\rangle$) is associated with spin-up (spin-down) in both terms. Hence we can simply denote the state by $|\alpha\rangle \otimes |\beta\rangle$ in which it is understood that the 'new' particle 1 – that associated with the first slot in the new notation – is spin-up and the new particle 2 – that associated with the second slot – is spin-down. So although the state is antisymmetric at a fundamental level, in this effective description we have two particles that are distinguished by their spins. Since the electrons are identical in the fundamental sense, and distinguished by constant properties in the effective description of this system, it would perhaps be more accurate to say, not that the electrons are approximately distinguishable, but that they are approximately non-identical.[7]

Second, while particles cannot be distinguished by continuously varying, exact positions, they can by continuously varying *approximate* positions. In the classical limit, identical particles have wavefunctions that are peaked in space with little overlap for some period; they are approximately trajectory distinguishable. Quantum mechanics does allow such states to evolve in a continuous way, with the peaks moving through space – as the existence of the classical limit demands. (And of course similar points hold for other observables.) If the particles in question are identical bosons or fermions, then these approximately distinct trajectories will serve to distinguish in just the way that spins did for the two electrons: we will be able to give an effective description of states in which the new ith slot is associated with the ith spatial trajectory. This is exactly what goes on for instance when we refer to an electron localized in a particular region of space, distinct from all other electrons.[8]

[7] Although the particles in the example of §1.1 are not fermions, they are – just for the evolution described – non-identical in a similar sense.

[8] Related issues in both classical and quantum mechanics are discussed in [5].

3 Why It Matters

Carefully distinguishing the concepts discussed in this article reveals a wider range of possibilities for multi-particle quantum systems, as is now briefly explained.

Messiah and Greenberg [6] were the first to exploit systematically the fact that the IP (which they called 'identicality'!) was not sufficient for the symmetrization postulate. Specifically, they relaxed the latter postulate and considered more general state spaces. Building on this work, Hartle, Stolt and Taylor (e.g., [7]) showed how to classify all types of identical, indistinguishable quantum particle statistics (compatible with a principle of 'cluster decompostion') according to the transformation properties of their state spaces under the action of particle permutations. However, they considered only observables satisfying the IP, which we have just seen to be an *ad hoc* restriction on observables. Thus, one may ask: 'Does also relaxing the IP allow an even richer classification of statistics by the transformation properties of states *and observables* under the action of particle permutations?'

And indeed it does, as Espinoza, Imbo and Satriawan [8] have recently shown. Bose and Fermi particles – what are usually called 'quanta' – are of course still examples of the types now classified, as are parastatistical particles and quantum Maxwell-Boltzmann particles, and a countable infinity of others. In every case categorized by Hartle, Stolt and Taylor (except for bosons and fermions which necessarily satisfy the indistinguishability postulate) there is an associated distinguishable case now possible in which non-symmetric observables are allowed. Any two systems with different statistics – whether they differ in the transformation properties of their states or observables or both – will have different partition functions and hence different thermodynamic behaviors. In particular, whether the indistinguishability postulate holds makes a real physical difference for a system of identical particles – or at least it would were we to discover identical yet distinguishable particles in nature.

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